

RELATION OF SQUEEZED STATES BETWEEN DAMPED HARMONIC AND SIMPLE HARMONIC OSCILLATORS

Chung-In Um

*Department of Physics, College of Science, Korea University,
Seoul 136-701, Korea*

Kyu-Hwang Yeon

*Department of Physics, Chungbuk National University, Cheong Ju,
Chung Buk 306-769, Korea*

Thomas F. George and Lakshmi N. Pandey

*Departments of Chemistry and Physics,
Washington State University, Pullman, Washington 99164-1046, USA*

Abstract

The minimum uncertainty and other relations are evaluated in the framework of the coherent states of the damped harmonic oscillator. It is shown that the coherent states of the damped harmonic oscillator are the squeezed coherent states of the simple harmonic oscillator. The unitary operator is also constructed, that connects coherent states between damped harmonic and simple harmonic oscillators.

1 Introduction

Recently there has been a surge of interest in the minimum uncertainty state which is one of the fundamental features of quantum mechanics[1]. Introducing the canonical conjugate variables for the harmonic oscillator, position x and momentum p in the appropriate dimensionless units, the coherent states can be described by a symmetric uncertainty in x and p with $\Delta p \cdot \Delta x = 1$ and $\Delta x = \Delta p = 1$. From the restriction of the uncertainty principle, $\Delta x \cdot \Delta p$, we may consider a more precise position $\Delta x < 1$ and a more uncertain momentum $\Delta p > 1$. These states, i.e., one variable is squeezed at the expense of its conjugate, are called squeezed states or minimum uncertainty states, which can not be obtained from the optical sources generating the coherent states[2], but from two-photon coherent states[3] including ordinary coherent states as a special case. This kind

of change in the variable corresponds to the measurement of either x or p in a rotating frame in phase space. This new space is the quadrature phase, that is directly related with a homodyne or heterodyne detection. Recently, two-photon devices have produced the squeezed states of light[4] with high precision interferometers[5].

The two-photon coherent states or minimum uncertainty can be distinguished from a coherent state in many ways, i.e., different photon processes, quantum statistical properties and coherence properties. The coherent state can be generated from one-photon stimulated processes, while the two-photon coherent states are generated from two-photon processes for two photons of the same mode. For the photon annihilation operator with frequency ω , we may define the coherent states $|\alpha\rangle$ ($a|\alpha\rangle = \alpha|\alpha\rangle$), and for the case of a two-photon process, a self-adjoint operator $a \equiv a_1 + ia_2$ yields $\langle \Delta a_1^2 \rangle = \langle \Delta a_2^2 \rangle = 1/4$ for the coherent state $|\alpha\rangle$, as derived in Sec. 3 below. However, the states with a more precise quantity $\langle \Delta a_1^2 \rangle \ll 1/4$ and a more uncertain $\langle \Delta a_2^2 \rangle \gg 1/4$ are permitted by the uncertainty $\langle \Delta a_1^2 \rangle \langle \Delta a_2^2 \rangle \geq 1/16$ with minimum uncertainty $\langle \Delta a_1^2 \rangle \langle \Delta a_2^2 \rangle = 1/16$. This indicates that the ordinary coherent states are different from the minimum uncertainty.

The purpose of this paper is to show that our previous results[6] of the coherent states for the damped harmonic oscillator (DHO) are the squeezed states of simple harmonic oscillator (SHO). Introducing the Caldirola-Kanai Hamiltonian[7], we review the propagator, wave function, uncertainty relation and coherent states[8] of the Caldirola-Kanai Hamiltonian in Sec. 2. In Sec. 3 we define the self-adjoint operator and construct the coherent states for DHO. We determine the properties and structure of the unitary transformation of the coherent states of DHO and SHO in Sec. 4. The results and discussion will be given in Sec. 5 together with graphs.

2 Propagator and Wave Function of DHO

We introduce the Caldirola-Kanai Hamiltonian for DHO as

$$\mathcal{H} = e^{-\gamma t} \frac{p^2}{2m} + e^{\gamma t} \frac{1}{2} m \omega_0^2 x^2, \quad (1)$$

where γ is the positive constant. As we have developed the quantum theory of damped driven harmonic oscillator by the path integral method[8], the propagator and wave function of DHO are given as

$$K(x, t; x_0, 0) = \left[\frac{m\omega e^{\frac{\gamma}{2}t}}{2\pi i \hbar \sin \omega t} \right]^{1/2} \exp \left[\frac{im}{4\hbar} \{ \gamma(x_0^2 - e^{\gamma t} x^2) + \frac{2\omega}{\sin \omega t} ((x^2 e^{\gamma t} + x_0^2) \cos \omega t - 2e^{\frac{\gamma}{2}t} x x_0) \} \right], \quad (2)$$

$$\Psi_n(x, t) = \frac{N}{(2^n n!)^{1/2}} H_n(Dx) \exp \left[-i(n + \frac{1}{2}) \cot^{-1} \left(\frac{\gamma}{2\omega} + \cot \omega t \right) - Ax^2 \right], \quad (3)$$

where

$$\omega = \left(\omega_0^2 - \frac{\gamma^2}{4} \right)^{1/2},$$

$$\begin{aligned}
N(t) &= \left[\frac{m\omega}{\pi\hbar} \right] \frac{e^{-\frac{\gamma}{2}t}}{\xi(t)^2}, \\
\xi(t)^2 &= \frac{\gamma^2}{4\omega^2} \sin^2 \omega t + \frac{\gamma}{2\omega} \sin 2\omega t + 1, \\
A(t) &= \frac{m\omega e^{\gamma t}}{2\hbar \xi(t)^2} \left\{ 1 + i \left[\left(\frac{\gamma^3}{8\omega^3} + \frac{\gamma}{\omega} \right) \sin^2 \omega t + \frac{\gamma^2}{8\omega^2} \sin 2\omega t \right] \right\}, \\
D(t) &= \left(\frac{m\omega}{\hbar} \right)^{1/2} \frac{e^{\frac{\gamma}{2}t}}{\xi(t)}.
\end{aligned} \tag{4}$$

To construct the coherent states ($|\alpha\rangle$) for DHO, we define the annihilation operator a and creation operator a^\dagger as

$$a = \frac{1}{i\hbar}(\eta x - \mu p), \tag{5}$$

$$a^\dagger = \frac{1}{i\hbar}(\mu^* p - \eta^* x), \tag{6}$$

where $\mu(t)$ and $\eta(t)$ are

$$\mu(t) = \frac{1}{2}(ReA)^{-1/2} \exp \left\{ i \cot^{-1} \left(\frac{\gamma}{2\omega} + \cot \omega t \right) \right\}, \tag{7}$$

$$\eta(t) = \sqrt{2i\hbar} \frac{A}{D} \exp \left\{ i \cot^{-1} \left(\frac{\gamma}{2\omega} + \cot \omega t \right) \right\}. \tag{8}$$

Equations (5)-(6) satisfy the commutation relation $[a, a^\dagger] = 1$, which corresponds to $[x, p] = i\hbar$. The coherent states in the coordinate representation $|x\rangle$ can be expressed by

$$\langle x | \alpha \rangle = (2\pi\mu\mu^*)^{-1/4} \exp \left[-\frac{1}{2i\hbar} \frac{\eta}{\mu} x^2 + \frac{\alpha}{\mu} x - \frac{1}{2} |\alpha|^2 - \frac{1}{2} \frac{\mu^*}{\mu} \alpha^2 \right]. \tag{9}$$

With the use of Eqs. (5)-(8) the uncertainty relation can be easily obtained as

$$\begin{aligned}
(\Delta x \Delta p) &= |\mu| |\eta| = \frac{\hbar}{2} \beta(t) \\
&= \frac{\hbar}{2} \left\{ 1 + \left[\left(\frac{\gamma^3}{8\omega^3} + \frac{\gamma}{\omega} \right) \sin^2 \omega t + \frac{\gamma^2}{8\omega^2} \sin 2\omega t \right]^2 \right\}^{1/2}
\end{aligned} \tag{10}$$

Here, Eq. (10) is the minimum uncertainty corresponding to the (0,0) states. All of the formulas derived above reduce to those of simple harmonic oscillator (SHO) when $\gamma = 0$. The propagator [Eq. (2)] has a very similar form to those of Cheng[9] and others[10], but Eq. (3) is of a new form.

3 Two-Dimensional Self-Adjoint Operators

We introduce the dimensionless two self-adjoint operators

$$a \equiv a_1 + ia_2, \quad a_1 = a_1^\dagger, \quad a_2 = a_2^\dagger \tag{11}$$

and the corresponding eigenstates

$$|\alpha\rangle = |\alpha_1\rangle_1 + i|\alpha_2\rangle_2, \quad (12)$$

where α_1 and α_2 are real. We refer to (a_1, a_2) or (α_1, α_2) as the quadrature components, and the relation between Eqs. (11) and (12) are given by

$$a_1 |\alpha_1\rangle_1 = \alpha_1 |\alpha_1\rangle_1, \quad (13)$$

$$a_2 |\alpha_2\rangle_2 = \alpha_2 |\alpha_2\rangle_2.$$

Using Eqs. (5)-(6) we may express Eq. (11) as

$$a_1 = \frac{1}{2i\hbar}[(\eta - \eta^*)x + (\mu^* - \mu)p], \quad (14)$$

$$a_2 = \frac{1}{2\hbar}[-(\eta + \eta^*)x + (\mu + \mu^*)p]. \quad (15)$$

Rewriting Eqs. (14) and (15) as the representation of x and p , we get

$$x = (\mu + \mu^*)a_1 - i(\mu - \mu^*)a_2, \quad (16)$$

$$p = (\eta + \eta^*)a_1 - i(\eta - \eta^*)a_2. \quad (17)$$

With the use of the wave function expressed as Eq. (3) and through the following definition

$$\langle \Delta a_1^2 \rangle_{m,n} \equiv \langle (a_1 - \langle a_1 \rangle_{m,n})(a_1 - \langle a_1 \rangle_{m,n})^* \rangle_{m,n}, \quad (18)$$

we obtain the uncertainty relations at various states as

$$\langle \Delta a_1^2 \rangle_{n+2,n} \langle \Delta a_2^2 \rangle_{n+2,n} = \frac{1}{16}(n+2)(n+1) \xrightarrow{\min.} \frac{1}{8}, \quad (19)$$

$$\langle \Delta a_1^2 \rangle_{n+1,n} \langle \Delta a_2^2 \rangle_{n+1,n} = \frac{1}{16}(n+1)^2 \xrightarrow{\min.} \frac{1}{16}. \quad (20)$$

$$\langle \Delta a_1^2 \rangle_{n,n} \langle \Delta a_2^2 \rangle_{n,n} = \frac{1}{16}(2n+1)^2 \xrightarrow{\min.} \frac{1}{16}, \quad (21)$$

$$\langle \Delta a_1^2 \rangle_{n-1,n} \langle \Delta a_2^2 \rangle_{n-1,n} = \frac{1}{16}n^2 \xrightarrow{\min.} \frac{1}{16}, \quad (22)$$

$$\langle \Delta a_1^2 \rangle_{n-2,n} \langle \Delta a_2^2 \rangle_{n-2,n} = \frac{1}{16}n(n-1) \xrightarrow{\min.} \frac{1}{8}. \quad (23)$$

Averages in the coherent states can be defined as

$$\langle \alpha | a | \alpha \rangle = \langle a \rangle = \alpha, \quad (24)$$

and thus we have

$$\langle a_1 \rangle = \frac{1}{2}(\alpha + \alpha^*) = \alpha_1, \quad (25)$$

$$\langle a_2 \rangle = \frac{i}{2}(\alpha^* - \alpha) = \alpha_2, \quad (26)$$

$$\langle a_1^\dagger a_1 \rangle = \alpha_1^2 + \frac{1}{4}, \quad (27)$$

$$\langle a_2^\dagger a_2 \rangle = \alpha_2^2 + \frac{1}{4}, \quad (28)$$

$$\langle \Delta a_1^2 \rangle = \langle \Delta a_2^2 \rangle = \frac{1}{4}, \quad (29)$$

and the following α_1 representation

$$\langle \alpha_1 | \alpha \rangle = \left(\frac{2}{\pi}\right)^{1/4} \exp \left[-(\alpha_1' - \alpha)^2 + \frac{i}{4} \alpha \operatorname{Im} \alpha \right], \quad (30)$$

where $a_1 | \alpha_1 \rangle = \alpha_1' | \alpha \rangle$.

4 Unitary Transformation

Now we will construct the unitary operator that transforms the coherent states for SHO to that of the two-photon coherent state of DHO and *vice versa*. From Eqs. (5)-(6), we can easily show the relation

$$a = \nu a_0 - \lambda a_0^\dagger, \quad (31)$$

$$a^\dagger = -\lambda^* a_0 + \nu^* a_0^\dagger, \quad (32)$$

where the expressions of a_0 and a_0^\dagger by a and a^\dagger are

$$a_0 = \nu^* a + \lambda a^\dagger, \quad (33)$$

$$a_0^\dagger = \lambda^* a + \nu a^\dagger, \quad (34)$$

for a pair of numbers λ and ν satisfying

$$|\nu|^2 - |\lambda|^2 = 1. \quad (35)$$

We take the values of ν and λ as the following:

$$\begin{aligned} \nu &= \sqrt{\frac{m\omega_0}{2\hbar}} \mu - i \frac{1}{\sqrt{2m\omega_0\hbar}} \eta \\ &= \frac{1}{2\xi} \sqrt{\frac{\omega}{\omega_0}} e^{\frac{\gamma}{2}} \left[\frac{\omega}{\omega_0} e^{-\gamma t} + (1 - i\sqrt{1-\beta^2}) \right] \exp \left[i \cot^{-1} \left(\frac{\gamma}{2\omega} + \cos \omega t \right) \right] \\ &= \frac{1}{2} \left[\sqrt{\frac{\omega^0}{\omega}} e^{-\frac{\gamma}{2}} \left(\frac{\gamma}{2\omega} \sin \omega t + \cos \omega t \right) - \sqrt{\frac{\omega}{\omega_0}} e^{\frac{\gamma}{2}} \left(\cos \omega t - \frac{\gamma}{2\omega} \sin \omega t \right) \right] \\ &\quad + \frac{i}{2} \left[\sqrt{\frac{\omega^0}{\omega}} e^{-\frac{\gamma}{2}} \sin \omega t - \sqrt{\frac{\omega}{\omega_0}} e^{\frac{\gamma}{2}} \left(\frac{\gamma^2}{4\omega^2} \sin \omega t + \sin \omega t \right) \right] \\ &= \frac{1}{2\xi} \left[\frac{\omega^0}{\omega} e^{-\gamma t} + 2 + \frac{\omega}{\omega_0} \beta^2 \right]^{1/2} \exp \left[i \cot^{-1} \left(\frac{\gamma}{2\omega} + \cot \omega t \right) + \tan^{-1} \left(\frac{\sqrt{1-\beta^2}}{\frac{\omega^0}{\omega} e^{-\gamma t} + 1} \right) \right], \end{aligned} \quad (36)$$

$$\begin{aligned}
\lambda &= \sqrt{\frac{m\omega_0}{2\hbar}}\mu + i\frac{1}{\sqrt{2m\omega_0\hbar}}\eta \\
&= \frac{1}{2\xi}\sqrt{\frac{\omega}{\omega_0}}e^{\frac{\gamma}{2}t}\left[\frac{\omega}{\omega_0}e^{-\gamma t} - (1 + i\sqrt{1-\beta^2})\right]\exp\left[i\cot^{-1}\left(\frac{\gamma}{2\omega} + \cot\omega t\right)\right] \\
&= \frac{1}{2}\left[\sqrt{\frac{\omega^0}{\omega}}e^{-\frac{\gamma}{2}t}\left(\frac{\gamma}{2\omega}\sin\omega t + \cos\omega t\right) - \sqrt{\frac{\omega}{\omega_0}}e^{\frac{\gamma}{2}t}\left(\cos\omega t - \frac{\gamma}{2\omega}\sin\omega t\right)\right] \\
&\quad + \frac{i}{2}\left[\sqrt{\frac{\omega^0}{\omega}}e^{-\frac{\gamma}{2}t}\sin\omega t - \sqrt{\frac{\omega}{\omega_0}}e^{\frac{\gamma}{2}t}\left(\frac{\gamma^2}{4\omega^2}\sin\omega t + \cos\omega t\right)\right] \\
&= \frac{1}{2\xi}\left[\frac{\omega_0}{\omega}e^{-\gamma t} - 2 + \frac{\omega}{\omega_0}\beta^2\right]^{1/2}\exp\left[i\cot^{-1}\left(\frac{\gamma}{2\omega} + \cot\omega t\right) + \tan^{-1}\left(\frac{\sqrt{1-\beta^2}}{\frac{\omega_0}{\omega}e^{-\gamma t} - 1}\right)\right].
\end{aligned} \tag{37}$$

Since a canonical transformation is defined as any transformation which keeps the commutator invariant, we can confirm that the transformation of variables from (a_0, a_0^\dagger) to (a, a^\dagger) given in Eqs. (31)-(32) and (35) is a canonical linear transformation. According to a theorem of Von Neumann[11], there exists a unitary operator U_α which yields all the linear canonical transformations, i.e.,

$$b(a_0, a_0^\dagger) = U_\alpha a_0 U_\alpha^\dagger = \nu a_0 - \lambda a_0^\dagger. \tag{38}$$

The commutation relation $[a, a^\dagger] = 1$ and unitary transformation [Eq. (38)] provide a with properties exactly similar to those of a_0 . Therefore, we may obtain the usual properties of a as

$$N = a^\dagger a, \tag{39}$$

$$N|n\rangle = n|n\rangle, \quad N|0\rangle = 0, \tag{40}$$

$$|n\rangle = U_\alpha |n\rangle_0, \tag{41}$$

and a coherent state for DHO is given by

$$|\alpha\rangle = U_\alpha |\alpha\rangle_0, \tag{42}$$

where $|\alpha\rangle_0$ is a coherent states for SHO. The representation of coherent states for DHO in the SHO space is given by

$$\langle\alpha|\alpha\rangle_0 = \frac{1}{\sqrt{\nu^*}}\exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\alpha_0|^2 + \frac{\lambda^*}{2\nu^*}\alpha_0^2 - \frac{\lambda}{2\nu^*} + \frac{1}{\nu^*}\alpha^*\alpha_0\right], \tag{43}$$

where the coefficients are

$$\begin{aligned}
\frac{1}{\sqrt{\nu^*}} &= \frac{-\sqrt{2\xi}}{\left(2 + \frac{\omega_0}{\omega}e^{-\frac{\gamma}{2}t} + \frac{\omega}{\omega_0}\beta^2\right)^{1/4}}, \\
\theta_\nu &= \tan^{-1}\left[\frac{\left(\frac{\gamma}{2\omega}\sin\omega t + \cos\omega t\right)\sqrt{\beta^2-1} + \left(\frac{\omega_0}{\omega}e^{-\gamma t} + 1\right)\sin\omega t}{\left(\beta\xi + \frac{\gamma}{2\omega}\sin\omega t + \cos\omega t\right)\left(\frac{\omega_0}{\omega}e^{-\gamma t} + 1\right) - \sin\omega t\sqrt{\beta^2-1}}\right], \\
\frac{\lambda^*}{2\nu^*} &= -\frac{1}{2}\left(\frac{\omega_0^2 e^{-\gamma t} - 2\omega\omega_0 + \omega^2\beta^2}{\omega_0^2 e^{-\gamma t} + 2\omega\omega_0 + \omega^2\beta^2}\right)^{1/2}\exp\left\{i\tan^{-1}\left[\frac{2\sqrt{\beta^2-1}}{2 - \beta^2 - \left(\frac{\omega_0}{\omega}\right)^2 e^{-2\gamma t}}\right]\right\},
\end{aligned} \tag{44}$$

$$\frac{\lambda}{2\nu^*} = -\frac{1}{2} \left(\frac{\omega_0^2 e^{-\gamma t} - 2\omega\omega_0 + \omega^2 \beta^2}{\omega_0^2 e^{-\gamma t} + 2\omega\omega_0 + \omega^2 \beta^2} \right)^{1/2} \exp \left\{ i \tan^{-1} \left[\frac{2\frac{\omega_0}{\omega} e^{-\gamma t} \sqrt{\beta^2 - 1}}{\beta^2 - (\frac{\omega_0}{\omega})^2 e^{-2\gamma t}} \right] \right\},$$

$$\frac{1}{\nu^*} = \frac{2\xi}{\left(2 + \frac{\omega_0}{\omega} e^{-\frac{\gamma t}{2}} + \frac{\omega}{\omega_0} \beta^2 \right)^{1/2}} e^{i2\theta_\nu}.$$

The wave function $\langle n | \alpha \rangle$ for a coherent state of DHO in the state of SHO can be obtained from Eq. (43). Using the following formula with the n th Hermite polynomial,

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n, \quad |t| < \infty, \quad (45)$$

and through the similar derivation of Eq. (9), we can easily obtain

$$\begin{aligned} {}_0 \langle n | \alpha \rangle &= \frac{1}{\sqrt{2^n n!}} \frac{\sqrt{2\xi}}{\left(\frac{\omega_0}{\omega} e^{-\gamma t} + 2 + \frac{\omega}{\omega_0} \beta^2 \right)} \left[\frac{\omega_0^2 e^{-\gamma t} - 2\omega\omega_0 + \omega^2 \beta^2}{\omega_0^2 e^{-\gamma t} + 2\omega\omega_0 + \omega^2 \beta^2} \right]^{\frac{n}{2}} \\ &\times H_n[(-2\nu\lambda)^{-1/2} \alpha] \exp\left(-\frac{1}{2} |\alpha|^2 - \frac{\lambda^*}{2\nu} \alpha^2\right), \end{aligned} \quad (46)$$

where the coefficients in Eq. (46) are given as

$$\begin{aligned} (-2\nu\lambda)^{-1/2} &= \frac{1}{2\xi} \left[\left(\frac{\omega_0}{\omega} e^{-\gamma t} + 2 + \frac{\omega}{\omega_0} \beta^2 \right) \left(\frac{\omega_0}{\omega} e^{-\gamma t} - 2 + \frac{\omega}{\omega_0} \beta^2 \right) \right]^{1/4} e^{i\theta_{\nu\lambda}}, \\ \theta_{\nu\lambda} &= \tan^{-1} \frac{\sqrt{\xi^2 - 1} + \sin \omega t}{\sqrt{\xi^2 - 1} - A_{\nu\lambda} \sin \omega t}, \\ A_{\nu\lambda} &= \frac{\sqrt{\left(\frac{\omega_0}{\omega} e^{-\gamma t} \right)^2 + 1 - \beta^2} - 2\frac{\omega_0}{\omega} e^{-\gamma t} + \sqrt{\left(\frac{\omega_0}{\omega} e^{-\gamma t} + 1 \right)^2 + 1 - \beta^2}}{B_{\nu\lambda}}, \\ B_{\nu\lambda} &= \left(\frac{\omega_0}{\omega} e^{-\gamma t} + 1 \right) \left(\frac{\omega_0}{\omega} e^{-\gamma t} - 1 \right) - \left(\frac{\omega_0}{\omega} e^{-\gamma t} + 1 \right)^2 \sqrt{\left(\frac{\omega_0}{\omega} e^{-\gamma t} - 1 \right)^2 + 1 - \beta^2} \\ &\quad - \left(\frac{\omega_0}{\omega} e^{-\gamma t} + 1 \right) \operatorname{sqr}t \left(\frac{\omega_0}{\omega} e^{-\gamma t} - 1 \right)^2 + 1 - \beta^2 \\ &\quad + \sqrt{\left[\left(\frac{\omega_0}{\omega} e^{-\gamma t} - 1 \right)^2 + 1 - \beta^2 \right] \left[\left(\frac{\omega_0}{\omega} e^{-\gamma t} + 1 \right)^2 + 1 - \beta^2 \right]}. \end{aligned} \quad (47)$$

If we represent the annihilation operator a_0 in the state of DHO, we get

$$\langle a_0 \rangle \equiv \langle \alpha | a_0 | \alpha \rangle = \nu^* \alpha + \lambda \alpha^* \equiv \alpha_{d1} + i\alpha_{d2}, \quad (48)$$

from the definition of Eq. (18) we obtain the quantities

$$\begin{aligned} \langle \Delta a_{01}^2 \rangle &\equiv \langle \alpha | (a_{01} - \alpha_{d1})^2 | \alpha \rangle \\ &= \frac{1}{4} |\nu + \lambda|^2 \\ &= \frac{m\omega_0}{2\hbar} |\mu|^2 \\ &= \frac{1}{4} \frac{\omega_0}{\omega} e^{-\gamma t} \xi^2, \end{aligned} \quad (49)$$

$$\begin{aligned}
\langle \Delta a_{02}^2 \rangle &\equiv \frac{1}{4} |\nu - \lambda|^2 \\
&= \frac{1}{2m\omega_0 \hbar} |\eta|^2 \\
&= \frac{1}{4} \frac{\omega}{\omega_0} e^{\gamma t} \frac{\beta^2}{\xi^2}, \tag{50}
\end{aligned}$$

$$\langle \Delta a_{01}^2 \rangle \langle \Delta a_{02}^2 \rangle = \frac{1}{16} \beta^2, \tag{51}$$

$$\begin{aligned}
\langle \Delta a_0^2 \rangle &= |\lambda|^2 \\
&= \frac{1}{2\xi} \left[\frac{\omega_0}{\omega} e^{-\gamma t} - 2 + \frac{\omega}{\omega_0} \beta^2 \right]^{1/2}. \tag{52}
\end{aligned}$$

The repetition of representation for the annihilation operator a in the state of SHO yields

$$\langle a \rangle_0 = \langle \alpha_0 | a | \alpha_0 \rangle = \nu \alpha_0 - \lambda \alpha_0^* \equiv \alpha_\lambda = \alpha_{\lambda 1} + i \alpha_{\lambda 2}, \tag{53}$$

$$\begin{aligned}
\langle \Delta a_1^2 \rangle_0 &\equiv \langle \alpha_0 | (a_1 - \alpha_{\lambda 1})^2 | \alpha_0 \rangle, \\
&= \frac{1}{4} |\nu^* - \lambda| \\
&= \frac{1}{4} \left[\frac{\omega_0}{\omega} e^{-\gamma t} \sin^2 \omega t + \frac{\omega}{\omega_0} e^{\gamma t} \left(\cos \omega t - \frac{\gamma}{2\omega} \sin \omega t \right)^2 \right], \tag{54}
\end{aligned}$$

$$\begin{aligned}
\langle \Delta a_2^2 \rangle_0 &= \frac{1}{4} |\nu^* + \lambda| \\
&= \frac{1}{4} \left[\frac{\omega_0}{\omega} e^{-\gamma t} \left(\frac{\gamma}{2\omega} \sin \omega t + \cos \omega t \right)^2 + \frac{\omega}{\omega_0} e^{\gamma t} \sin^2 \omega t \left(\frac{\gamma^2}{4\omega^2} + 1 \right) \right], \tag{55}
\end{aligned}$$

$$\begin{aligned}
\langle \Delta a_1^2 \rangle_0 \langle \Delta a_2^2 \rangle_0 &= \frac{1}{16} \left\{ \left[\frac{\omega_0}{\omega} e^{-\gamma t} \left(\frac{\gamma}{2\omega} \sin \omega t + \cos \omega t \right)^2 \right. \right. \\
&\quad + \frac{\omega}{\omega_0} e^{\gamma t} \left(\cos \omega t - \frac{\gamma}{2\omega} \sin \omega t \right) \left(\frac{\gamma^2}{2\omega^2} + 1 \right) \sin \omega t \Big]^2 \\
&\quad \left. + \left[\cos 2\omega t - \frac{\gamma^2}{2\omega^2} \sin^2 \omega t \right]^2 \right\}, \tag{56}
\end{aligned}$$

$$\begin{aligned}
\langle \Delta a^2 \rangle_0 &= |\lambda|^2 \\
&= \frac{1}{2\xi} \left(\frac{\omega_0}{\omega} e^{-\gamma t} - 2 + \frac{\omega}{\omega_0} \beta^2 \right)^2. \tag{57}
\end{aligned}$$

In Eq. (42) we have defined the unitary operator that is a linear canonical transformation. From this equation we have

$$\begin{aligned}
\langle \alpha_0 | \beta \rangle &= \langle \alpha_0 | U_L | \beta_0 \rangle \\
&= U_L^*(\beta_0, \alpha_0^*) \langle \alpha_0 | \beta \rangle. \tag{58}
\end{aligned}$$

A direct application of the following formulas[12]

$$e^{c_1 A} e^{c_2 B} e^{c_3 C} = \exp(c_1 A + c_2 B + c_3 C + \frac{1}{2}\{c_1 c_2 [A, B] + c_1 c_3 [A, C] + c_2 c_3 [B, C]\}) \\ + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \overbrace{[A, [A, \dots, [A, [B, [B, \dots, [B, C] \dots]]]}^i \overbrace{\frac{c_1^i c_2^j c_3}{(i+1)!(j+1)!}}^j \quad (59)$$

$$e^{c_1 a^{\dagger 2}} e^{c_2 a^{\dagger} a} e^{c_3 a^2} = \exp\{c_1 a^{\dagger 2} + c_2 a^{\dagger} a + c_3 a^2 - (a^{\dagger 2} + a^{\dagger} a + a a^{\dagger} + a^2) \\ + \sum_{j=1}^{\infty} \frac{(-2j)^j}{(j+1)!} [-2c_1 c_3 (a^{\dagger} a + a a^{\dagger}) + 8c_1^2 c_3 a^{\dagger 2}]\} \\ = \exp[(c_1 - 8c_1^2 c_3 \frac{e^{2c_1} - 1}{2c_2} - c_1 c_2) a^{\dagger 2} (c_2 - c_1 c_3 + 2c_1 c_3 \frac{e^{-2c_2} - 1}{2c_2}) a^{\dagger} a \quad (60) \\ + (-c_1 c_3 + 2c_1 c_3 \frac{e^{-2c_2} - 1}{2c_2}) a a^{\dagger} + (c_1 - c_2 c_3) a^2]$$

gives the unitary operator in the $|\alpha >_0$ representation,

$$U_L^{(n)}(\alpha_0, \alpha_0^*) = \frac{1}{\sqrt{\nu}} \exp \left[\frac{\lambda}{2\nu} \alpha_0^2 + \left(\frac{1}{\nu} - 1 \right) \alpha_0^* \alpha_0 - \frac{\lambda^*}{2\nu} \alpha_0^2 \right] \quad (61)$$

$$= \frac{1}{\sqrt{\nu\pi}} \exp \left[\frac{\lambda}{2\nu} a_0^{\dagger 2} - \ln \nu a_0^{\dagger} a_0 - \frac{\lambda^*}{2\nu} a_0^2 \right] \quad (62)$$

$$= \frac{1}{\sqrt{\nu\pi}} \exp [A_u a_0^{\dagger 2} + B_u a_0^{\dagger} a_0 + C_u a_0 a_0^{\dagger} + D_u a_0^2] , \quad (63)$$

where the coefficients are

$$A_u = \frac{\lambda}{2\nu} + \frac{\lambda}{2\nu} \ln \nu + \frac{2}{3} \left(\frac{\lambda}{2\nu} \right)^2 \left(\frac{\lambda^*}{2\nu} \right) \frac{1 - \nu^2}{\ln \nu} , \\ B_u = -\ln \nu - \frac{\frac{\lambda}{2\nu} \frac{\lambda^*}{2\nu}}{2 \ln \nu} (1 - \nu^2) , \\ C_u = -\frac{\frac{\lambda}{2\nu} \frac{\lambda^*}{2\nu}}{2 \ln \nu} (1 - \nu^2) , \\ D_u = \frac{\frac{\lambda}{2\nu}}{2 \ln \nu} (1 - \nu^2) . \quad (64)$$

5 Results and Discussion

Starting from the coherent states of DHO, we have shown that these states are the squeezed states of SHO and *vice versa*. We have also evaluated the averages of the operators $a_0, a, \Delta a_1^2$ and Δa_2^2 in both spaces of DHO and SHO. We have constructed the unitary operator which transforms the

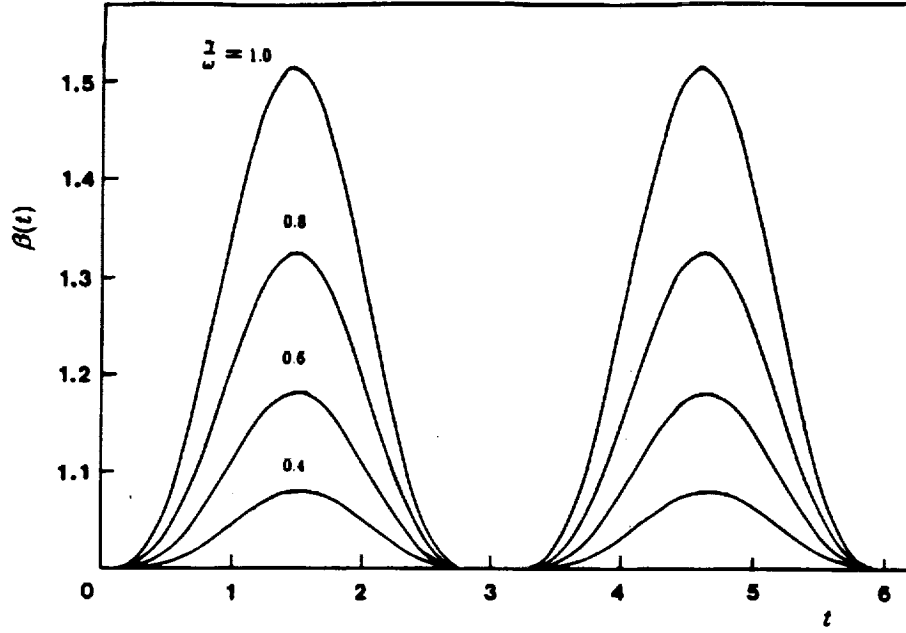


FIG. 1. $\beta(t)$ as a function of ωt at various values of $x = \gamma/\omega$.

coherent states ($|\alpha >_0$) to the coherent states ($|\alpha >$), i.e., $|\alpha > = U_\alpha |\alpha >_0$.

Figure 1 illustrates the behavior of $\beta(t)$ [Eq. (10)] as a function of t and $x = \gamma/\omega$. As x increases, the amplitude of the oscillation becomes large. For the condition $\gamma \ll \omega_0$, $\omega \simeq \omega_0$ and $\gamma \rightarrow 0$, $\beta(t)$ approaches to unity, with DHO reducing to SHO. Therefore, the uncertainty relation for the (n,n) state [Eq. (10)] oscillates with the period π .

From the definition of the self adjoint operator and Eq. (18), we have evaluated the minimum uncertainty for various states in Eqs. (19)-(23). The minimum uncertainties for the diagonal and first off-diagonal states have the value of $1/16$, and the minimum values for the second off-diagonal states are $1/8$. For $\langle \Delta a_1^2 \rangle \ll 1/4$, the corresponding canonical part results in more uncertainty.

The creation and annihilation operators (a^\dagger and a) in Sec. 4 can be shown under the condition $|\nu|^2 - |\lambda|^2 = 1$. The operators (a_0^\dagger, a_0) are transformed to the operators (a^\dagger, a) through unitary operator U_α . The behaviors of $|\nu|$ and $|\lambda|$ are depicted in Figures 2 and 3, respectively. We can confirm that $|\nu|$ oscillates periodically in general, but $|\lambda|$ behaves in a more complicated fashion, and as $x = \gamma/\omega$ increases to larger than unity, the oscillation decays rapidly.

The average of Δa_{01}^2 and Δa_{02}^2 in the states of DHO are given in Eqs. (49)-(50). $\langle \Delta a_{01}^2 \rangle$ oscillates with exponential decrease, while $\langle \Delta a_{02}^2 \rangle$ does so with exponential increase. The minimum value of $\langle \Delta a_{01}^2 \rangle \langle \Delta a_{02}^2 \rangle$ is $1/16$ at $\beta(t) = 1$. The averages of Δa_1^2 and Δa_2^2 in the space of SHO are evaluated in Eqs. (54)-(55). The uncertainty relation [Eq. (56)] has a minimum value of $1/16$ at $t = \sin^{-1} n\pi$ or $t = \cos^{-1}(\gamma/2\omega - 4\omega/\gamma)$, and maximum value at $\beta^2 = (\omega_0/\omega)^2 e^{-2\gamma t}$ (Figure 4).

Equations (61)-(63) represent the unitary operator which transforms $|\alpha >_0$ to $|\alpha >$ and vice versa. Therefore, we can obtain the scaled state through $\langle x | \alpha > = \langle x | U | \alpha >_0$.

In conclusion, we have shown the uncertainties and their relations in the states of SHO and

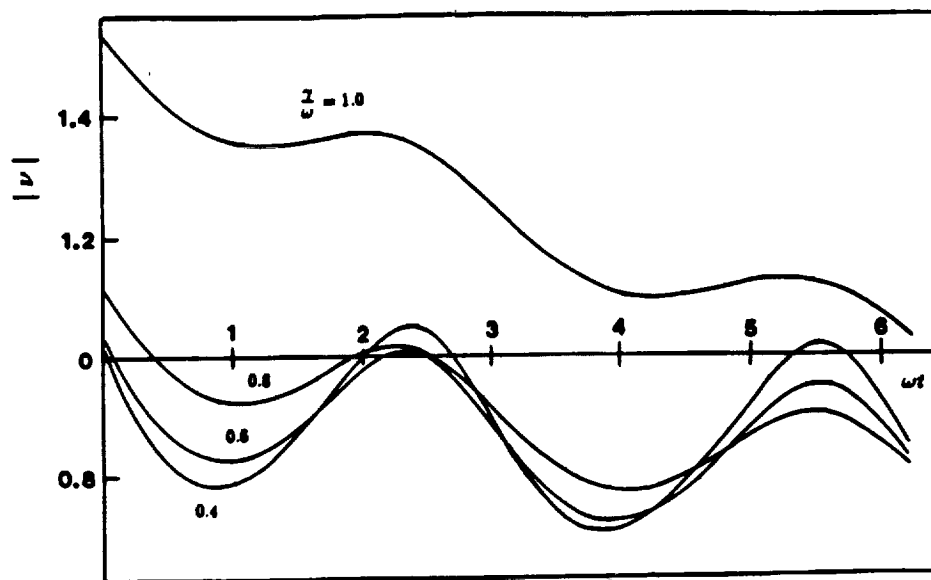


FIG. 2. $|\nu|$ versus ωt at various values of γ/ω .

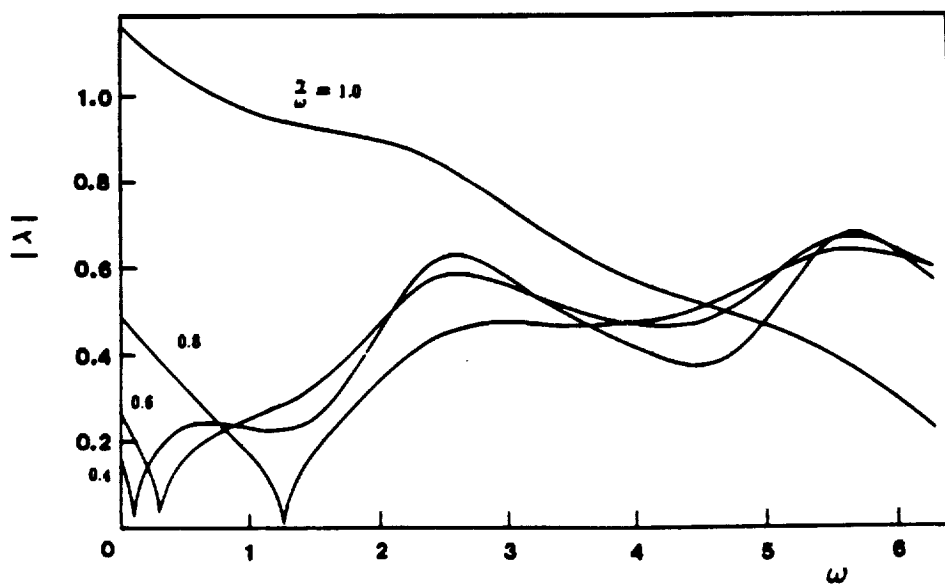


FIG. 3. $|\lambda|$ versus ωt at various values of γ/ω .

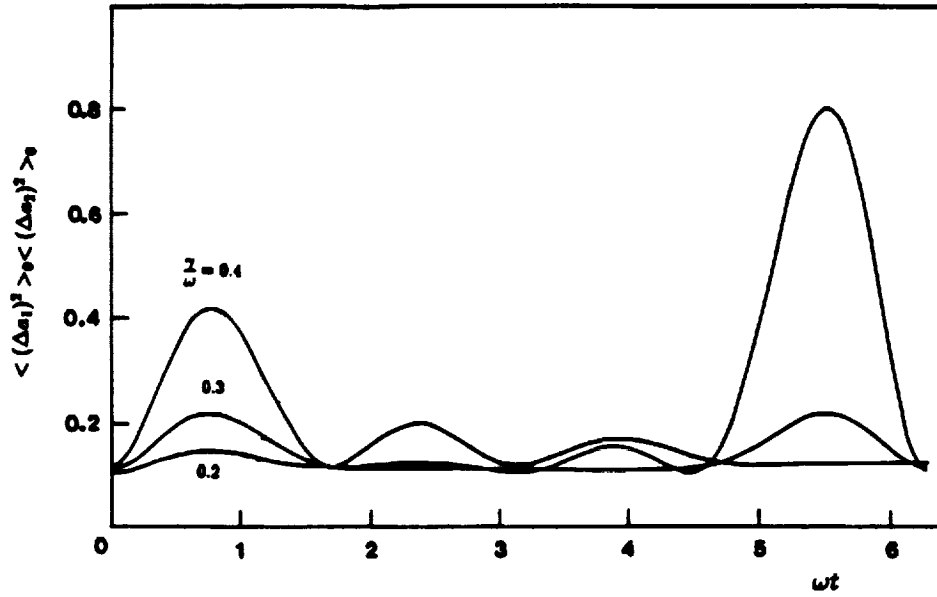


FIG. 4. Uncertainty relations versus ωt .

DHO. We have also shown that there exists a unitary operator to connect the coherent states of SHO with those of DHO.

6 Acknowledgments

This research was supported by the Center for Thermal and Statistical Physics, BSRI Program, Ministry of Education 1992, and the National Science Foundation under Grant CHE-9196214.

References

- [1] E. H. Kennard, Z. Phys. **44**, 326 (1927) ; D. Stoler, Phys. Rev. D **1**, 3217 (1970) ; **4**, 1925 (1971) ; Phys. Rev. Lett. **23**, 1397 (1974) ; H. P. Yuen, Phys. Lett. A **51**, 1 (1975).
- [2] R. J. Glauber, Phys. Rev. **131**, 2766 (1963) ; Phys. Rev. Lett. **10**, 84 (1963).
- [3] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976) ; V. Bužek, A. Vidiella-Barranco and P. L. Knight, Phys. Rev. A **45**, 6570 (1992).
- [4] A. L. Robinson, Science **233**, 280 (1986) ; R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz and J. F. Valley, Phys. Rev. Lett. **55**, 2409 (1985).
- [5] B. Yurke, J. R. Klauder and S. L. McCall, Phys. Rev. D **33**, 4033 (1986) ; R. S. Bondurant and J. H. Shapiro, Phys. Rev. D **30**, 2548 (1984).

- [6] K. H. Yeon, C. I. Um and T. F. George, *Phys. Rev. A* **36**, 5287 (1987).
- [7] P. Caldirola, *Nuovo Cimento* **18**, 393 (1941) ; **B 77**, 241 (1983) ; E. Kanai, *Prog. Theor. Phys.* **3**, 440 (1940).
- [8] C. I. Um, K. H. Yeon, and W. H. Kahng, *J. Phys. A: Math. Gen.* **20**, 611 (1987) ; H. G. Oh, H. R. Lee, T. F. George and C. I. Um, *Phys. Rev. A* **39**, 5515 (1989).
- [9] B. K. Cheng, *J. Phys. A* **17**, 2475 (1985).
- [10] D. C. Khandekar and S.V. Lawanda, *J. Math. Phys.* **16**, 384 (1975).
- [11] J. Von Neumann, *Math. Ann.* **104**, 570 (1931).
- [12] William H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973), Chap. 3, pp. 132-229; G. J. Papadopoulos and J. T. Devreese, Eds., *Path Integral and Their Applications in Quantum, Statistical and Solid State Physics*, NATO Advanced Study Institute Series (Plenum, New York, 1978), pp. 85-162.

